CHAPTER ONE

1. INTRODUCTION

1.1. Definitions and scope of operations research

1.1.1. Definition

Operations Research (OR) is a science which deals with problem identification, formulation, solutions and finally appropriate decision making.

OR is the use of mathematical models, statistics and algorithm to aid in decision-making. It is most often used to analyze complex real life problems typically with the goal of improving or optimizing performance.

"OR is concerned with scientifically deciding how to best design and operate man-machine system usually requiring the allocation of scare resources."

"OR is essentially a collection of mathematical techniques and tools which in conjunction with system approach, are applied to solve practical decision problems of an economic or engineering nature".

1.1.2. Scope of operations research

A common misconception held by many is that OR is a collection of mathematical tools. While it is true that it uses a variety of mathematical techniques, operations research has a much broader scope. The following are some of the areas operations research has been practiced. These are:

Human resource management

- To appoint the suitable persons on minimum salary
- To determine the best age of retirement

Production

- To find the number and size of the items to be produced
- Optimum product mix
- Select, locate and design product mix

Finance

- To maximize per capita income with minimum resources
- Find out profit plan for the company
- Best replacement policies

Marketing

- Assignment of sales men for the territory
- Travelling sales man problem

1.2. Development of operation research

Since the advent of the industrial revolution, the world has seen a remarkable growth in the size and complexity of organizations. An integral part of this revolutionary change has been a tremendous increase in the division of labor and segmentation of management responsibilities in these organizations. So, this increasing specialization has created new problems, problems that are still occurring in many organizations. A related problem is that as the complexity and specialization in an organization increase, it becomes more and more difficult to allocate the available resources to the various activities in a way that is most effective for the organization as a whole. These kinds of problems and the need to find a better way to solve them provided the environment for the emergence of **operations research** (commonly referred to as **OR**).

The roots of OR can be traced back many decades, when early attempts were made to use a scientific approach in the management of organizations. However, the beginning of the activity called operations research has generally been attributed to the military services early in World War II. Because of the war effort, there was an urgent need to allocate scarce resources to the various military operations and to the activities within each operation in an effective manner. Therefore, the British and then the U.S. military management called upon a large number of scientists to apply a scientific approach to dealing with this and other strategic and tactical problems. In effect, they were asked to do research on (military) operations. These teams of scientists were the first OR teams. By developing effective methods of using the new tool of radar, these teams were instrumental in winning the Air Battle of Britain. Through their research on how to better manage group and antisubmarine operations, they also played a major role in winning the Battle of the North Atlantic. Similar efforts assisted the Island Campaign in the Pacific.

When the war ended, the success of OR in the war effort spurred interest in applying OR outside the military as well. As the industrial boom following the war was running its course, the problems caused by the increasing complexity and specialization in organizations were again coming to the forefront. It was becoming apparent to a growing number of people, including business consultants who had served on or with the OR teams during the war, that these were basically the same problems that had been faced by the military but in a different context. By the early 1950s, these individuals had introduced the use of OR to a variety of organizations in business, industry, and government. The rapid spread of OR soon followed.

Operations Research has existed since the beginning of recorded history. As far back as World War II, operations research techniques have been developed to assist the Military during the war. Today, many organizations employ a staff of operation researcher or management science personnel or consultants to apply the principles of operations research to management problems.

Pre-World war II: The roots of OR are as old as science and society. Though the roots of OR extend to even early 1800s, it was in 1885 when Ferderick W. Taylor emphasized the application of scientific analysis to methods of production, that the real start took place. Another man of early scientific management era was Henry L. Gantt. Most job scheduling methods at that time were rather random. A job, for instance, may be processed on a machine without dilemma but then wait for days for acceptance by the next machine. Gantt mapped each job from machine to machine, minimizing every delay. Now with the Gantt procedure, it is possible to plan machine loadings months in advance and still quote delivery dates accurately.

However, it was the First Industrial Revolution which contributed mainly towards the development of OR. Before this revolution, most of the industries were small scale, employing only a handful of men. The advent of machine tools-the replacement of man by machine as a source of power and improved means of transportation and communication resulted in fast flourishing industry. It became increasingly difficult for a single man to perform all the managerial functions (of planning, sale, purchase, production, etc.). Consequently, a division of management function took place. Managers of production, marketing, finance, personnel, research and development etc., began to appear. With further industrial growth, further subdivisions of management functions took place. For example, production department was sub-divided into sections like maintenance, quality control, procurement, production planning, etc.

Through this science of operation research originated in England, the united states soon took the lead in united states these OR teams helped in developing strategies from mining operations, inventing new flight patterns and planning of sea mines.

Post-world war II: immediately after the war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their problems. Industrial operation research in U.K. and U.S.A. developed along different lines. In U.K., the critical economic situation required drastic increase in production efficiency and creation of new markets. Nationalization of a few key industries further increased the potential field for OR. Consequently, OR soon spread from military to government, industrial, social and economic planning.

The progress of operational research in U.S.A. was due to advent of second industrial revolution which resulted in automation-the replacement of man by machine as a source of control. The new revolution began around 1940s when electronic computers became commercially available. In 1950, OR was introduced as a subject for academic study in American universities since then this subject has been gaining ever increasing importance for the students of Mathematics, Statistics, Commerce, Economics, Management and Engineering.

1.3 Overview of the operations research modeling approach

Modeling approach for problem solving in operation research needs the following steps:

Defining the problem and gathering data

This procedure is crucial. It is difficult to extract a "right" answer from the "wrong" problem.
Most practical problems encountered by OR teams are initially described in a vague and imprecise way. This step should answer the following questions: Who are the decision makers? , What are the objectives? , What are the constraints (relationships)? and how to collect relevant data.

Formulating a mathematical model

After the decision maker's problem is defined, the next phase is to reformulate this problem in a form that is convenient for analysis. The conventional OR approach for doing this is to construct a mathematical model that represents the essence of the problem. Before discussing how to formulate such a model, we first explore the nature of models in general and of mathematical models in particular. Mathematical models are also idealized representations, but they are expressed in terms of mathematical symbols and expressions. Construct a mathematical model that represents the essence of the problem. This step needs to define decision variables, define the objective function, and define the constraints (relations among decision variables).

A crucial step in formulating an OR model is the construction of the objective function. This requires developing a quantitative measure of performance relative to each of the decision maker's ultimate objectives that were identified while the problem was being defined. If there are multiple objectives, their respective measures commonly are then transformed and combined into a composite measure, called the **overall measure of performance.** This overall measure might be something tangible (e.g., profit) corresponding to a higher goal of the organization, or it might be abstract (e.g., utility). In the latter case, the task of developing this measure tends to be a complex one requiring a careful comparison of the objectives and their relative importance. After the overall measure of performance is developed, the objective function is then obtained by expressing this measure as a mathematical function of the decision variables.

Deriving solutions from the model

After a mathematical model is formulated for the problem under consideration, the next phase in an OR study is to develop a procedure (usually a computer-based procedure) for deriving solutions to the problem from this model. Sometimes, in fact, it is a relatively simple step, in which one of the standard **algorithms** (systematic solution procedures) of OR is applied on a computer by using one of a number of readily available software packages. For experienced OR practitioners, finding a solution is the fun part, whereas the real work comes in the preceding and following steps, including the post optimality analysis (sensitivity analysis).

A common theme in OR is the search for an **optimal,** or best, **solution.** Indeed, many procedures have been developed for finding such solutions for certain kinds of problems. However, it needs to be recognized that these solutions are optimal only with respect to the model being used. Since the model necessarily is an idealized rather than an exact representation of the real problem. This step needs an algorithm (systematic solution procedures) and post-optimality analysis (what would happen to the optimal solution if different assumptions are made?)

Implementation

Install a well-documented system for applying the model. Include the model, solution procedure, and operating procedures for implementation. This system is usually computer-based. A considerable number of computer programs often need to be used and integrated. Databases and management information systems may provide up-to-date input for the model. The assumptions of the model continue to be satisfied. Need to revise or re-build models when significant deviations occur.

1.4. Applications of operations research

- It provides a tool for scientific analysis and provides solution for various business problems.
- It enables optimum allocation of scarce resources.
- It helps in minimizing waiting and servicing costs.
- It enables the management to decide when to buy and how much to buy through the technique of inventory planning.
- It helps in evaluating situations involving uncertainty.
- It enables experimentation with models, thus eliminating the cost of making errors while experimenting with reality.
- It allows quick and inexpensive examination of large numbers of alternatives.
- In general, OR facilitates and improves the decision making process.

CHAPTER TWO
LINEAR PROGRAMMING

2.1. Basic Concepts in Linear Programming

Linear programming has as its purpose the optimal allocation of scarce resources among competing products or activities. It is singularly helpful in business and economics where it is often necessary to optimize a profit or cost function subject to several inequality constraints. If the constraints are limited to two variables, the easiest solution is through the use of graphs.

Linear programming is a mathematical technique for finding optimal solutions to problems that can be expressed using linear equations and inequalities. If a real-world problem can be represented accurately by the mathematical equations of a linear program, the method will find the best solution to the problem. Of course, few complex real-world problems can be expressed perfectly in terms of a set of linear functions. Nevertheless, linear programs can provide reasonably realistic representations of many real-world problems — especially if a little creativity is applied in the mathematical formulation of the problem.

A linear program consists of a set of variables, a linear objective function indicating the contribution of each variable to the desired outcome, and a set of linear constraints describing the limits on the values of the variables.

2.2. Formulation of Linear Programming Problems

Similarly, the mathematical model of a business problem is the system of equations and related mathematical expressions that describe the essence of the problem. Thus, if there are n related quantifiable decisions to be made, they are represented as **decision variables** (say, $x1$, $x2$, . . . , xn) whose respective values are to be determined. The appropriate measure of performance (e.g., profit) is then expressed as a mathematical function of these decision variables (for example, $P = 3x1 + 2x2 +\ldots+5xn$). This function is called the **objective function.** Any restrictions on the values that can be assigned to these decision variables are also expressed mathematically, typically by means of inequalities or equations. Such mathematical expressions for the restrictions often are called **constraints.** The constants (namely, the coefficients and right-hand sides) in the constraints and the objective function are called the **parameters** of the model. The mathematical model might then say that the problem is to choose the values of the decision variables so as to maximize the objective function, subject to the specified constraints. Such a model, and minor variations of it, typifies the models used in OR. The objective of a linear programming problem will be to maximize or to minimize some numerical value.

Objective Function: Objective function is the goal or objective of a management, stated as intent to maximize or to minimize some important quantity such as profits or costs.

The objective function takes the following general form:

$$z = \sum_{i=1}^{n} c_i x_i$$

Where:

C_i = the objective function coefficient corresponding to the i^{th} variable, and

X_i = the i^{th} decision variable

The coefficients of the objective function indicate the contribution to the value of the objective function of one unit of the corresponding variable.

Decision Variables: The Variables in a linear program are a set of quantities that need to be determined in order to solve the problem; i.e., the problem is solved when the best values of the variables have been identified. Typically, the variables represent the amount of a resource to use or the level of some activity. Simply, things which produced sold or consumed.

Constraints define the possible values that the variables of a linear programming problem may take. Factors that affect the production, sale and consumption of goods and services. They typically represent resource constraints, or the minimum or maximum level of some activity or condition. They take the following general form:

Subject to $\sum_{i=1}^{n} a_i x_i < b_i$

x_i = the i^{th} decision variable

a_i = the coefficient on Xi in constraint and

bi = the right-hand-side coefficient on constraint

Non-negativity Constraints: the variables of linear programs must always take non-negative values (i.e., they must be greater than or equal to zero).

Finally LPP model is represented as follows:

$$\boldsymbol{z = \sum_{i=1}^{n} c_i x_i}$$

Subject to $\sum_{i=1}^{n} a_i x_i < b_i$

$X_i > 0$

In formulating the LPP as a mathematical model, we shall follow the following four steps

1. Identify the decision variables and assign symbols to them (eg x, y, z,. . .or x_1, x_2, x_3, . . .
2. Identify the set if constraints and express them in terms of inequalities involving the decision variables.
3. Identify the objective function and express it in terms of the decision variables.
4. Add the non-negativity constraints

2.3. Assumptions of linear programming

Before we get too focused on solving linear programs, it is important to review some theory For instance, several assumptions are implicit in linear programming problems. These assumptions are:

1. Proportionality: deals with the contribution per unit of each decision variable to the objective function
2. Optimization: Objective function to maximize or minimize
3. Finiteness: A finite number of activities and constraints to consider
4. Determinism: All parameters are assumed to be known constants
5. Continuity: All decision variables should be continuous variables.
6. Homogeneity: All units of the same resource or activity are identical
7. Additivity: When two or more activities are used, the total product is equal to the sum of the individual products (no interaction effects between activities)

2.4. Methods of Solving LP

2.4.1. The Graphic Method

To sole LPP graphically, the following steps are necessary:

1. Formulate mathematical model of Linear programming
2. Convert constraint inequalities into equalities
3. Draw the graph by intercept
4. Identify the feasible area of the solution which satisfies all constrains.
5. Identify the corner points in the feasible region
6. Identify the optimal point
7. Interpret the result

❖ **Graphical LP is a two-dimensional model.**

Maximization Problem

==>Maximize Z with inequalities of constraints in $\leq$ form

Example: Consider two models of color TV sets; **Model A** and **B**, are produced by a company to maximize profit. The profit realized is $300 from A and $250 from set B. The limitations are

a. Availability of only 40hrs of labor each day in the production department.
b. a daily availability of only 45 hrs on machine time
c. Ability to sale 12 set of model A.

How many sets of each model will be produced each day so that the total profit will be as large as possible?

Resources used per unit

Constraints	**Model A (X1)**	**Model B (X2)**	**Maximum Available hrs.**
Labor hr.	2	1	40
Machine hr.	1	3	45
Marketing hr.	1	0	12
Profit	**$300**	**$250**	

Solution

1. Formulation of mathematical modeling of LPP

Max $Z=300X_1+250X_2$

St: $2X_1+X_2 \leq 40$

$X_1+3X_2 \leq 45$

$X_1 \quad \leq 12$

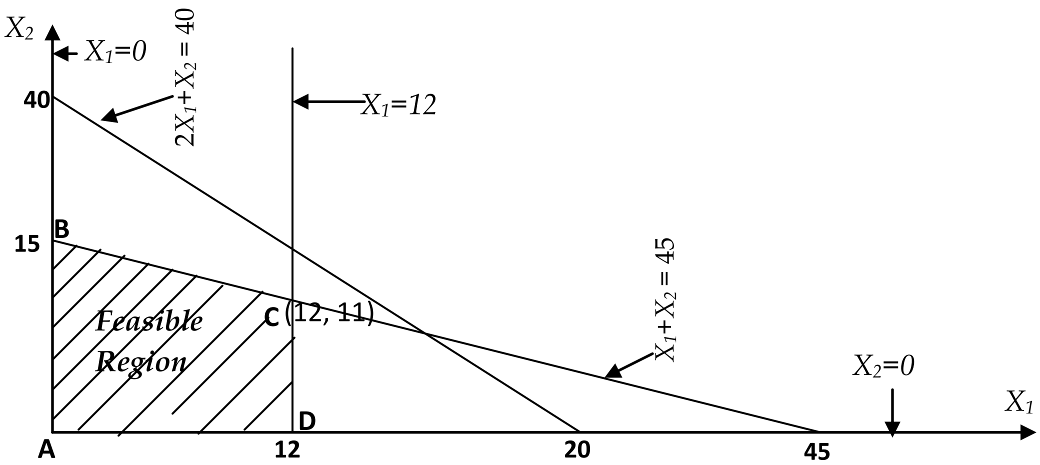
$X_1, X_2 \quad \geq 0$

LPP Model

2. Convert constraints inequalities into equalities

$2X_1+X_2 = 40$
$X_1+3X_2 = 45$
$X_1 \quad = 12$

3. Draw the graph by intercepts
$2X_1+X_2 = 40$ ==> (0, 40) and (20, 0)
$X_1+3X_2 = 45$==> (0, 15) and (45, 0)
$X_1 \quad = 12$==> (12, 0)
X_1 X2 = 0



X_2
$X_1=0$
40
$2X_1+X_2=40$
$X_1=12$
B
15
Feasible
Region
C (12, 11)
$X_1+X_2=45$
$X_2=0$
X_1
D
A
12
20
45

4. Identify the feasible area of the solution which satisfies all constrains.
5. Identify the corner points in the feasible region
   A (0, 0), B (0, 15), C (12, 11) and D (12, 0)
6. Identify the optimal point
7. Interpret the result

Corners	Coordinates	Max Z=300 X_1 +250X_2
A	(0, 0)	\$0
B	(0, 15)	\$3750
C	**(12, 11)**	**\$6350**
D	(12, 0)	\$3600

Interpretation: 12 units of model A and 11 units of model B should be produced so that the total profit will be \$6350.

B. Minimization Problem

==>Minimize Z with inequalities of constraints in≥ form

Example: Suppose that a machine shop has two different types of machines; machine 1 and machine 2, which can be used to make a single product .These machines vary in the amount of product produced per hr., in the amount of labor used and in the cost of operation.

Assume that at least a certain amount of product must be produced and that we would like to utilize at least the regular labor force. How much should we utilize each machine in order to utilize total costs and still meets the requirement?

	Resource used		
Constraints	**Machine 1 (X_1)**	**Machine (X_2)**	Minimum required hours
Product produced/hr	20	15	100
Labor/hr	2	3	15
Operation Cost	**\$25**	**\$30**	

Solution

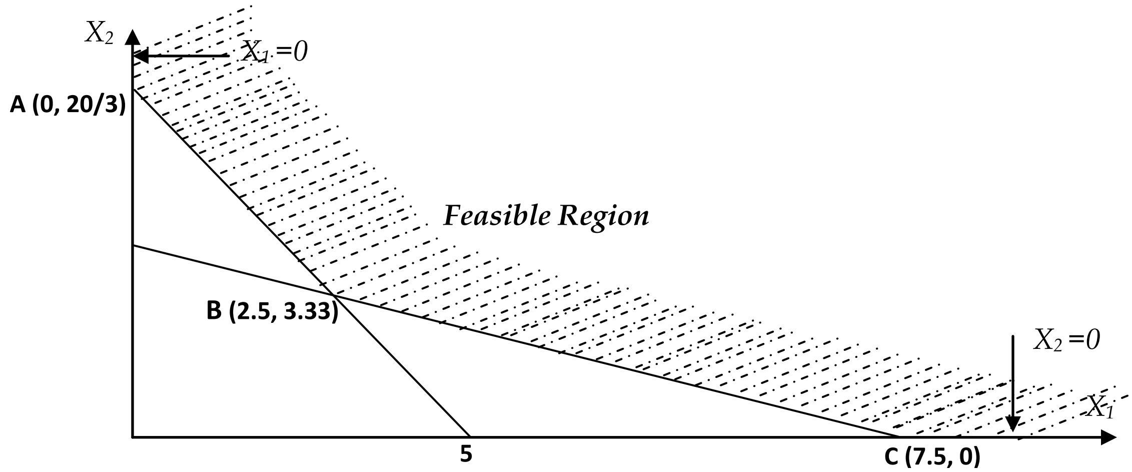
$$
\left.\begin{aligned}
&Min.Z = 25X_1+30X_2\\
&St:\\
&20X_1+15X_2\geq 100\\
&2X_1+3X_2\geq 15\\
&X_1, X_2 \geq 0
\end{aligned}\right\} \text{LPP Model}
$$

Constraint equation:

$20X_1 + 15X_2 = 100$ ==> (0, 20/3) and (5, 0)

$2X_1+3X_2=15$ ==> (0, 5) and (7.5, 0)

$X_1\ X_2 = 0$



Corners	Coordinates	MinZ=25 X_1 + 30X_2
A	(0, 20/3)	200
B	**(2.5, 3.33)**	**162.5**
C	(7.5, 0)	187.5

$\mathbf{X_1}$ =2.5unit

$\mathbf{X_2}$=3.33unit and

MinZ= **$162.5**

SPECIAL CASES IN GRAPHICS METHODS

1. Redundant Constraint

If a constraint when plotted on a graph doesn't form part of the boundary making the feasible region of the problem that constraint is said to be redundant constraint.

Example: A firm is engaged in producing two products A and B .Each unit of product A requires 2Kg of raw material and 4 labor-hours for processing; whereas each unit of product B requires 3Kg of raw materials and 3hrs of labor. Every unit of product A needs 4hrs to packaging and every unit of product B needs 3.5hrs for packaging. Every week the firm has availability of 60Kg of raw material, 96 labor-hours and 105 hrs. I the packaging department.1 unit of product A sold yields $40 profit and 1 unit of B sod yields $35 profit.

Required:

a. Formulate this problem as a LPP

b. Find the optimal solution

Solution

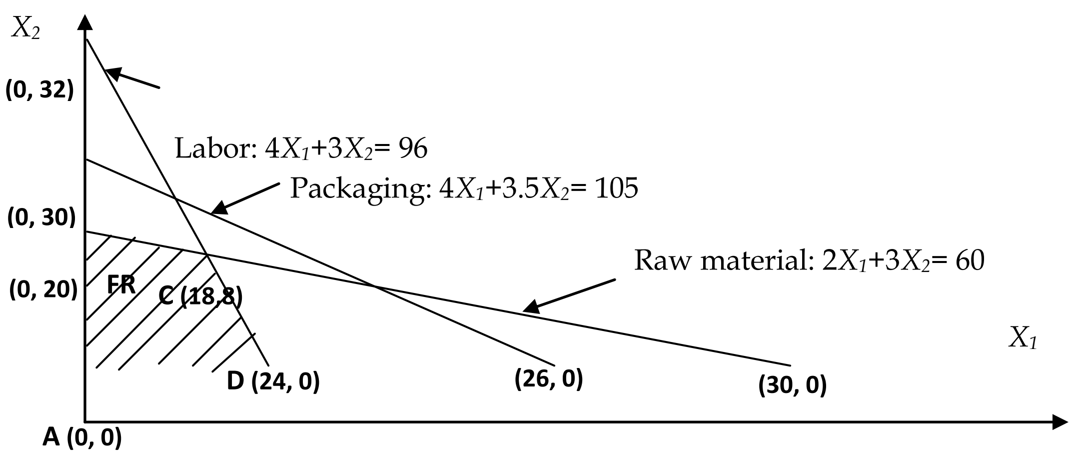
	Products A	Products B	Resource available Resources per week
Raw materials (Kg)	2	3	60
Labor (hr)	4	3	96
Packaging (hr)	4	3.5	105
Profit per unit	**$40**	**$35**	

Let $\mathbf{X_1}$ =The N$^{\underline{o}}$of units of product A produced per week

$\mathbf{X_2}$ =The N$^{\underline{o}}$of units of product B produced per week

a. LPP Model

$$Max.Z = 40X_1 + 35X_2$$
$$St:$$
$$2X_1 + 3X_2 \leq 60$$
$$4X_1 + 3X_2 \leq 96$$
$$4X_1 + 3.5X_2 \leq 105$$
$$X_1, X_2 \geq 0$$



❖ The packaging hr is redundant.

Corners	Coordinates	MinZ=40 X_1 + 35X_2
A	(0, 0)	0
B	(0, 20)	700
C	**(18, 8)**	**1000**
D	(24, 0)	960

X_1 =18

X_2=8 and

MinZ= **1000**

Interpretation:

The company should produce and sale 18 units of product A and 8 units of product B per week so as to get a maximum profit of 1000.

Note:

The packaging hour's constraint does not form part of the boundary making the feasible region. Thus, this constraint is of no consequence and is therefore, redundant. The inclusion or exclusion of a redundant constraint does not affect the **optimal solution** of the problem.

2. Multiple optimal Solutions

==>We have unlimited number of optimal solution without increasing or decreasing the objective function.

Example:

$$Max.Z = 8X_1 + 16X_2$$

$$St:$$

$$3X_1 + 6X_2 \leq 900$$

$$X_1 + X_2 \leq 200$$

$$X_2 \leq 125$$

$$X_1, X_2 \geq 0$$

==>Multiple optimal solutions provide more choices for management to reach their objectives.

3. Infeasible Solution

A solution is called feasible if it satisfies all the constraints and the constraints and non-negativity condition. However, it is sometimes possible that the constraints may be inconsistent so that there is no feasible solution to the problem. Such a situation is called **infeasibility.**

Example:

$MaxZ=20X_1+30X_2$

$St: 2X_1+X_2 \leq 40$

$4X_1+X_2 \leq 60$

$X_1 \geq 30$

$X_1, X_2 \geq 0$

4. Unbounded Solution It is a solution whose objective function is infinite. If the feasible region is unbounded then one or more decision variables will increase indefinitely without violating feasibility, and the value of the objective function can be made arbitrarily large. Consider the following model:

Example: **$Minz = 40x_1 + 60x_2$**

St: $2x_1 + x_2 \geq 70$

$x_1 + x_2 \geq 40$

$x_1 + 3x_2 \geq 90$

$x_1, x_2 \geq 0$

SIMPLEX METHOD

The graphical method is useful only for problems involving two decision variables and relatively few problem constraints. **What happens when we need more decision variables and more constraints?** We use an algebraic method called the simplex method, which was developed by George B. DANTZIG (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

The simplex method is an **ITERATIVE** or "**step by step**" method or repetitive algebraic approach that moves automatically from one basic feasible solution to another basic feasible solution improving the situation each time until the optimal solution is reached at.

MAXIMIZATION PROBLEMS

- Maximize Z with inequalities of constraints in "$\leq$ "form

Example: Solve the problem using the simplex approach

$\text{Max.}Z = 300x_1 + 250x_2$

Subject to: $2x_1 + x_2 \leq 40$ (Labor)

$x_1 + 3x_2 \leq 45$ (Machine)

$x_1 \leq 12$ (Marketing)

$x_1, \ x_2 \geq 0$

Solution

Step 1: Formulate LPP Model

Step 2: Standardize the problem

I.e. convert constraint inequality into equality form by introducing a variable called **Slack variable**.

Slack Variables: Slack variables represent the unused resources between the left-hand side and right-hand side of each inequality. A slack variable(s) is added to the left hand side of a ≤ constraint to covert the constraint inequality in to equality. The value of the slack variable shows unused resource.

A slake variable emerges when the LPP is a maximization problem. Slack variables represent unused resource or idle capacity. Thus, they don't produce any product and their contribution to profit is **zero**. Slack variables are added to the objective function with **zero coefficients**.

Let that s_1, s_2, and s_3 are unused labor, machine and marketing hrs respectively.

$$
\left.
\begin{aligned}
&\textbf{Max.Z} = 300x_1 + 250x_2 + 0s_1 + 0s_2 + 0s_3 \\
&\text{St:} 2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 40 \\
&x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 45 \\
&x_1 + \quad 0s_1 + 0s_2 + s_3 = 12 \\
&x_1, x_2, \quad s_1, \quad s_2, \quad s_3 \geq 0
\end{aligned}
\right\} \textit{Standard form}
$$

Step 3: Obtain the initial simplex tableau

To represent the data, the simplex method uses a table called the **simplex table** or the **simplex matrix**.

==> In constructing the initial simplex tableau, the search for of the optimal solution begins at the origin. Indicating that nothing can be produced;

Thus, first assumption, No production implies that $\mathbf{x_1=0}$ and $\mathbf{x_2=0}$

==>$2x_1+x_2+s_1+0s_2+0s_3=40$ ==>$x_1+3x_2+0s_1+s_2+0s_3=45$

$2(0)+0+s_1+0s_2+0s_3=40$ $0+3(0)+0s_1+s_2+0s_3=45$

$\underline{\underline{s_1=40}}$ – **Unused labor hrs.** $\underline{\underline{s_2=45}}$ – **Unused machine hrs.**

==>$x_1+0s_1+0s_2+s_3=12$

$0+0s_1+0s_2+s_3=12$

$\underline{\underline{s_3=12}}$ – **Unused Marketing hrs.**

Therefore, **Max.Z**$=300x_1+250x_2+0s_1+0s_2+0s_3$

$=300(0)+250(0)+0(40)+0(45)+0(12) = \underline{\underline{0}}$

Note: In general, whenever there are **n** variables and **m** constraints (excluding the non-negativity), where **m** is less than **n** (**m<n**), **n-m** variables must be **set equal to zero** before the solution can be solved algebraically.

a. Basic variables are variables with non-zero solution values.
   Or: basic variables are variables that are in the basic solution. Basic variables have **0** values in the $C_j - Z_j$ row.
b. Non-basic variables are variables with zero solution values.
   Or: non-basic variables are variables that are out of the solution.

==>**n=5** variables (x_1, x_2, s_1, s_2, and s_3) and **M=3** constraints (Labor, machine and marketing constraints), excluding non-negativity.

Therefore, **n-m**=5-3=2 variables(x_1 and x_2) are set equal to zero in the 1st simplex tableau. These are non-basic variables. 3 Variables (s_1, s_2, and s_3) are basic variables (in the 1st simplex tableau) because they have non-zero solution values.

Step 4: Construct the initial simplex tableau

Initial simplex tableau

Cj		300	250	0	0	0		
	SV	X1	X2	S1	S2	S3	Q	
0	S1	2	1	1	0	0	40	***R₁***
0	S2	1	3	0	1	0	45	***R₂***
0	S3	1	0	0	0	1	12	***R₃***
	Zj	0	0	0	0	0	0	
	Cj - Zj	300	250	0	0	0		

Step 5: Choose the "incoming" or "entering" variables

Note: The entering variable is the variable that has the most positive value in the **Cj – Zj** row also called as indicator row. Or the entering variable is the variable that has the highest contribution to profit per unit. X1 in our case is the entering variable. The column associated with the entering variable is called key or pivot column (X1columnin our case)

Pivot Column: The column of the tableau representing the variable to be entered into the solution mix.

Step 6: Choose the "leaving "or "outgoing" variable

==> In this step, we determine the variable that will leave the solution for X1 (or entering variable)

Note: The row with the minimum or lowest positive (non-negative) replacement ratio shows the variable to leave the solution. Pivot Row: The row of the tableau representing the variable to be replaced in the solution mix.

Replacment Ratio (RR) $= \frac{\text{solution Quantity(Q) (RHS)}}{\text{Corresponding Values in Pivot Column}}$

Note: RR>0

The variable leaving the solution is called leaving variable or outgoing variable.

The row associated with the leaving variable is called key or pivot row (s3 columnin our case)

The element that lies at the intersection of the pivot column and pivot row is called pivot element(No 1 in our case). Pivot Number: The element in both the pivot column and the pivot row.

Step 7: Repeat step 4-6 till optimum basic feasible solution is obtained.

Or: repeat step 3-5 till no positive value occurs in the Cj - Zj row.

2nd simplex tableau

Cj		300	250	0	0	0	
	SV	X1	X2	S1	S2	S3	Q
0	S1	0	1	1	0	-2	16
0	S2	0	3	0	1	-1	33
300	X1	1	0	0	0	1	12
	Zj	300	0	0	0	300	3600
	Cj - Zj	0	250	0	0	-300	

$R'_1=R_1-2R_3$

$R'_2=R_2-R_3$

$R'_3=\underline{R_3}$

3rd simplex tableau

Cj		300	250	0	0	0	
	SV	X1	X2	S1	S2	S3	Q
0	S1	0	0	1	-1/3	-5/3	5
250	X2	0	1	0	1/3	-1/3	11
300	X1	1	0	0	0	1	12
	Zj	300	250	0	250/3	650/3	6350
	Cj - Zj	0	0	0	-250/3	- 650/3	

Since all the Cj - Zj< 0 optimal solution is reached at.

Therefore, X1=12, X2=11, S1=5 and Max Z=6350

MINIMIZATION PROBLEMS

- Minimize Z with inequalities of constraints in "$\geq$ "form
  There are two methods to solve minimization LP problems:

1. Direct method/Big M-method/

    - Using artificial variables

2. Conversion method

    - Minimization by maximizing the dual

- **Surplus Variable (-s):**
    - A variable inserted in a greater than or equal to constraint to create equality. It represents the amount of resource usage above the minimum required usage.
    - Surplus variable is subtracted from a $\geq$ constraint in the process of converting the constraint to standard form.
    - Neither the slack nor the surplus is negative value. They must be positive or zero.

Thus, in order to avoid the mathematical contradiction, we have to add artificial variable (A)

- **Artificial variable** (**A**):

Artificial variable is a variable that has no meaning in a physical sense but acts as a tool to create an initial feasible LP solution.

1. Big M-method /Charnes Penalty Method/

In this method; we assign coefficients to artificial variables, undesirable from the objective function point of view. If objective function Z is to be minimized, then a very large positive price (called **penalty**) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called **penalty**) is assigned to each of these variables.

Following are the characteristics of Big-M Method:

a. High penalty cost (or profit) is assumed as M
b. M is assigned to artificial variable A in the objective function Z.
c. Big-M method can be applied to minimization as well as maximization problems with the following distinctions:
    i. **Minimization problems**
        - Assign **+M** as coefficient of artificial variable A in the objective function **Z**
    ii. **Maximization problems:**
        - Here**– M** is assigned as coefficient of artificial variable **A** in the objective function **Z**
d. Coefficient of **S** (slack/surplus) takes zero values in the objective function **Z**
e. For minimization problem, the incoming variable corresponds to the **highest negative** value of $\mathbf{C_j - Z_j}$.
f. Solution is optimal when there is no negative value of $\mathbf{C_j\text{-}Z_j}$.(For minimization case)

Example: Min $Z=25x_1 + 30x_2$

Subject to: $20x_1+15x_2 \geq 100$

$2x_1+ 3x_2 \quad \geq 15$

$x_1, \quad x_2 \geq 0$

Solution

Step 1 Standardize the problem

Minimize $Z=25x_1 + 30x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

Subject to:

$20x_1+15x_2 - s_1 + A_1 = 100$

$2x_1 + \quad 3x_2 - s_2 + A_2 \ = 15$

$x_1, \ x_2 \ , s_1, s_2 , A_1 , A_2 \geq 0$

Step 2: Initial simplex tableau

The initial basic feasible solution is obtained by setting $x_1= x_2= s_1= s_2=0$

No production, $x_1= x_2= s_1=0 ==> 20(0) + 15(0) - 0 + A_1 = 100 ==> A_1 = 100$

$x_1= x_2= s_2=0 ==> 0(0)+3(0) - 0 + A_2 = 15 ==> A_2 = 15$

Initial simplex tableau

C_j		25	30	0	0	M	M	
	SV	X_1	X_2	S_1	S_2	A_1	A_2	Q
M	A_1	20	15	-1	0	1	0	100
M	A_2	2	3	0	-1	0	1	15
	Z_j	22M	18M	-M	-M	M	M	115 M
	C_j - Z_j	25 -22M	30- 18	M	M	0	0	

Note: Once an **artificial variable** has left the basis, it has served its purpose and can therefore be removed from the simplex tableau. An artificial variable is never considered for re-entry into the basis.

2nd Simplex Tableau

C_j		25	30	0	0	M	
	SV	**X_1**	**X_2**	**S_1**	**S_2**	**A_2**	**Q**
25	**X_1**	1	3/4	-1/20	0	0	5
M	**A_2**	0	3/2	1/10	-1	1	5
	Z_j	**25**	75/4+**3/2M**	-5/4+1/10**M**	**-M**	**M**	**125+5 M**
	C_j - Z_j	**0**	**45/4-3/2M**	5/4-1/10 **M**	**M**	**0**	

R'_1=R_1/20

3[rd] Simplex Tableau

C_j - $Z_j \geq$ 0==>Optimal solution is reached

X_1=5/2

X_2=10/3 and MinZ=162.5

Note: As long as an "**A**" variable is available in the solution variable column, the solution is **infeasible.**

R''_1=R'_1-3/4R''_2

C_j		25	30	0	0	
	SV	**X_1**	**X_2**	**S_1**	**S_2**	**Q**
25	**X_1**	1	0	-1/10	1/2	5/2
30	**X_2**	0	1	1/15	-2/3	10/3
	Z_j	**25**	30	-1/2	-15/2	**162.5**
	C_j - Z_j	**0**	**0**	1/2	**15/2**	

SPECIAL CASES IN SIMPLEX METHOD

1. Mixed constraints

Example

Max $Z=6x_1 +8x_2$

Subject to: $x_2 \leq 4$

$6x_1+ 2x_2 \geq 24$

$x_1, \ x_2 \geq 0$

- **Standard form**

Max.Z$=6\mathbf{x_1} +8\mathbf{x_2} + 0\,\mathbf{s_1} +0\,\mathbf{s_2}+ 0\,\mathbf{s_3} - M\,\mathbf{A_2} - M\,\mathbf{A_3}$

St: $\mathbf{x_2}+ \mathbf{s_1} = 4$

$6x_1+2x_2 - \mathbf{s_3} + \mathbf{A_3} =24$

All Variables ≥ 0

2. Two incoming variables

In order to break this tie, the selection for the key column (entering variable) can be made arbitrary. However; the number of solution can be minimized by adopting the following rules:

1. If there is a tie between two decision variables, then the selection can be made arbitrary.

2. If there is a tie between a decision variable and a slack (or surplus) variable, then select the decision variable to enter into basis first.

3. If there is a tie between slack or surplus variable, then selection can be made arbitrary.

3. Infeasibility

In the simplex method, an infeasible solution is indicated by looking at the final tableau .In it, all Cj - Zj row entries will be the proper sign to imply optimality, but an artificial variable (A) will still be in the solution mix.

Example: Minimization case

C_j		5	8	0	0	M	
	SV	**X_1**	**X_2**	**S_1**	**S_2**	**A_2**	**Q**
5	**X_1**	1	1	-2	3	0	200
8	**X_2**	0	1	1	2	0	100
M	**A_2**	0	0	0	-1	1	20
	Z_j	5	**8**	**-2**	**31-M**	**M**	1,800+200M
	C_j - Z_j	**0**	**0**	**2**	**M-31**	**0**	

Even though all **C_j - Z_j** are positive or 0(i.e the criterion for an optimal solution in a minimization case), **no feasible solution** is possible because **an artificial variable (A_2)** remains in the solution mix.

4. Unbounded Solutions

No finite solution may exist in problems that are not bounded .This means that a variable can be infinitely large without violating a constraint. In the simplex method, the condition of unboundedness will be discovered prior to reaching the final tableau. We will note the problem when trying to decide which variable to remove from the solution mix. The procedure in unbounded solution is to divide each quantity column number by the corresponding pivot column number. The row with the smallest **positive** ratio is replaced. But if the entire ratios turn out to be **negative** or **undefined**, it indicates that the problem is unbounded. **Example:** Maximization case

C_j		6	9	0	0	
	SV	**X_1**	**X_2**	**S_1**	**S_2**	**Q**
9	**X_2**	-1	1	2	0	30
0	**S_2**	-2	0	-1	1	10
	Z_j	-9	**9**	**18**	**0**	270
	C_j - Z_j	**15**	**0**	**-18**	**0**	

The solution in the above case is not optimal because not all **C_j - Z_j** entries are **0** or negative, as required in a maximization problem. The next variable to enter the solution should be **X_1**.To determine which variable will leave the solution, we examine the ratios of the quantity column numbers to their corresponding numbers in the **X_1**or pivot column. Since both pivot column numbers are negative, an unbounded solution is indicated.

- No unbounded solutions, no outgoing variable will exist.

5. Degeneracy

C_j		5	8	2	0	0	0	
	SV	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	**Q**
8	**X_2**	1/4	1	1	-2	0	0	10
0	**S_2**	4	0	1/3	-1	1	0	20
0	**S_3**	2	0	2	2/5	0	1	10
	Z_j	2	**8**	**8**	**16**	**0**	**0**	80
	C_j - Z_j	**3**	**0**	**-6**	**-16**	**0**	**0**	

If there is a tie for the smallest ratio, this is a signal that degeneracy exists. Degeneracy can occur right in the first (initial tableau). This normally happens when the number of constraints is less than the number of variables in the objective function. Problem can be overcome by **trial and error method**. Degeneracy is shown in the simplex algorithm alternatives back and forth between the same non-optimal solutions, i.e., it puts a new variable in, then takes it out in the next tableau, puts it back in ,and so on.

One simple way of dealing with the issue is to select either row (**S_2 or S_3** in this case) arbitrary. If we are unlucky and cycling does occur, we simply go back and select the other row.

6. Multiple Optimal Solutions

Multiple optimal solutions exist when non-basic variable contains **zero** on its C_j **-** Z_j row.

Example: Maximization problem

Max $Z=3X_1+2X_2$

$X_1=0$, $X_2=6$, $S_2=3$ **and Max** $Z=12$ **or:** $X_1=3$, $X_2=3/2$ **and Max** $Z=12$

The C_j **-** Z_j value of the Non-basic variable (X_1) is 0.Thus, there is alternative optimal solution.

C_j		3	2	0	0	
	SV	X_1	X_2	S_1	S_2	Q
2	X_2	3/2	1	1	0	6
0	S_2	1	0	1/2	1	3
	Z_j	3	2	2	0	12
	$C_j - Z_j$	0	0	-2	0	